ДЗ 7

№ 1

$x \in R, \ y > 0$

$\Delta u = 0$

$$u(x,0) = \begin{cases} 1 & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases}$$

$$f(z_0, z) = \frac{z - z_0}{z - \overline{z_0}}$$

$$G(z_0, z) = \frac{1}{2\pi} \ln \frac{1}{|f|}$$

$$u(x_0, y_0) = -\int_{\Gamma} \frac{\partial G}{\partial n}\Big|_{\Gamma} \psi|_{\Gamma} \, d\Gamma$$

$$\frac{\partial G}{\partial n} = -\frac{1}{\pi} \frac{y_0}{(x - x_0)^2 + y_0^2}$$ – считали в классе

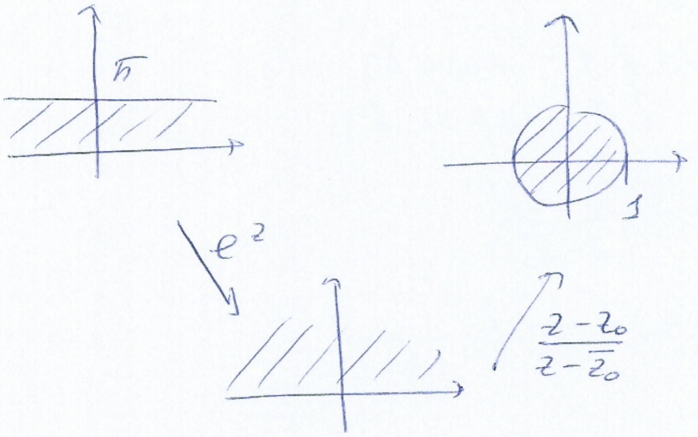
$$u(x_0, y_0) = \int_{-\infty}^{+\infty} \frac{y_0 \psi(x)}{\pi((x - x_0)^2 + y_0^2)} dx = \int_a^b \frac{y_0}{\pi((x - x_0)^2 + y_0^2)} dx =$$

$$= \frac{1}{\pi} \int_a^b \frac{d\left(\frac{x}{y_0}\right)}{\left(\frac{x - x_0}{y_0}\right)^2 + 1^2} = \frac{1}{\pi} \operatorname{arctg}\left(\frac{x - x_0}{y_0}\right)\Big|_a^b = \frac{1}{\pi} \operatorname{arctg}\left(\frac{b - x_0}{y_0}\right) - \frac{1}{\pi} \operatorname{arctg}\left(\frac{a - x_0}{y_0}\right)$$

Дз 7

№ 2

$$\begin{cases} \Delta u = 0 & x \in R \\ u(x,0) = 1 & 0 < y < \pi \\ u(x,\pi) = 0 \end{cases}$$



$$f(z_0, z) = \frac{e^z - e^{z_0}}{e^z - e^{\bar{z}_0}} = \frac{e^z - e^{z_0}}{e^z - e^{\bar{z}_0}}$$

$$|f(z_0,z)| = \left| \frac{e^z - e^{z_0}}{e^z - e^{\bar{z}_0}} \right| = \frac{A_1}{A_2}$$

$$A_1 = |e^z - e^{z_0}| = |e^{x+iy} - e^{x_0+iy_0}| = |e^x \cos y + ie^x \sin y - e^{x_0}\cos y_0 - ie^{x_0}\sin y_0| =$$

$$= \left((e^x \cos y - e^{x_0}\cos y_0)^2 + (e^x \sin y - e^{x_0}\sin y_0)^2 \right)^{1/2} = (e^{2x} + e^{2x_0} - 2e^x e^{x_0}\cos y \cos y_0 -$$

$$- 2e^x e^{x_0} \sin y \sin y_0)^{1/2} = (e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos(y - y_0))^{1/2}$$

$$A_2 = (e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos(y+y_0))^{1/2}$$

$$G = \frac{1}{2\pi} \ln \frac{1}{|f(z_0,z)|} = -\frac{1}{2\pi} \ln |f(z_0,z)|$$

Рассм $\frac{\partial G}{\partial n}$ на части границы $\Gamma_1 : y = 0$ (при $y=\pi$ интеграл обнулится за счет $u(x,\pi) = 0$)

$$\left.\frac{\partial G}{\partial n}\right|_{\Gamma_1} = \left.\frac{\partial G}{\partial(-y)}\right|_{y=0} = \left.\frac{1}{2\pi}\frac{\partial \ln|f(z_0,z)|}{\partial y}\right|_{y=0} = \left.\frac{1}{2\pi|f(z_0,z)|}\cdot |f(z_0,z)|'\right|_{y=0} =$$

$$= \left.\frac{A_2}{2\pi A_1}\cdot\left(\frac{A_1}{A_2}\right)'\right|_{y=0} = \left.\frac{A_2}{2\pi A_1}\,\frac{A_1' \cdot A_2 - A_1 \cdot A_2'}{A_2^2}\right|_{y=0} = \left.\frac{A_1' \cdot A_2 - A_1 \cdot A_2'}{2\pi A_1 \cdot A_2}\right|_{y=0}$$

$$A_1' = \frac{1}{2}\cdot\frac{1}{A_1}\cdot 2e^{x+x_0}\sin(y-y_0) \qquad A_2' = \frac{1}{2}\cdot\frac{1}{A_2}\cdot 2e^{x+x_0}\sin(y_0+y_0)$$

$$A_1\big|_{y=0} = (e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos y_0)^{1/2} = B = A_2\big|_{y=0}, \quad A_1'\big|_{y=0} = -\frac{e^{x+x_0}\sin y_0}{B} = \frac{C}{B} = -A_2'\big|_{y=0}$$

$$\Rightarrow \left.\frac{\partial G}{\partial n}\right|_{\Gamma_1} = \frac{\frac{C}{B}\cdot B - B\cdot\frac{-C}{B}}{2\pi B^2} = \frac{C}{\pi B^2} = -\frac{e^{x+x_0}\sin y_0}{\pi(e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos y_0)}$$

см продолжение

№2 (продолжение) Дз 7

$$u(x_0,y_0) = -\int_\Gamma \frac{\partial G}{\partial n}\Big|_\Gamma \psi\, d\Gamma = -\int_{\Gamma_1=\{x\in R,\ y=0\}} \frac{\partial G}{\partial n}\Big|_{\Gamma_1} \psi\Big|_{\Gamma_1} d\Gamma - \int_{\Gamma_2=\{x\in R,\ y=\pi\}} \frac{\partial G}{\partial n}\Big|_{\Gamma_2} \psi\Big|_{\Gamma_2} d\Gamma =$$

$$\left\{ \psi\Big|_{\Gamma_2} = u(x,\pi) = 0 \right\}$$

$$= -\int_{-\infty}^{+\infty} \frac{\partial G}{\partial(-y)}\Big|_{y=0} u(x,y)\Big|_{y=0} dx = -\int_{-\infty}^{+\infty} \left(-\frac{e^{x+x_0}\sin y_0}{\pi(e^{2x}+e^{2x_0}-2e^{x+x_0}\cos y_0)}\right)\cdot 1\, dx =$$

$$= \left\{ \begin{matrix} e^x = t \\ e^x dx = dt \\ x=-\infty \quad t=0 \\ x=+\infty \quad t=+\infty \end{matrix} \right\} = \int_0^{+\infty} \frac{e^{x_0}\sin y_0}{\pi(t^2+e^{2x_0}-2e^{x_0}t\cos y_0)}\, dt = \left\{ \begin{matrix} b = e^{x_0}\cos y_0 \\ c = e^{x_0} \\ |c|\geq|b| \end{matrix} \right\}$$

$$= \frac{e^{x_0}\sin y_0}{\pi}\int_0^{+\infty} \frac{dt}{t^2-2bt+c^2} = \frac{e^{x_0}\sin y_0}{\pi}\int_0^{\infty} \frac{dt}{(t-b)^2+\underbrace{c^2-b^2}_{\geq 0}} =$$

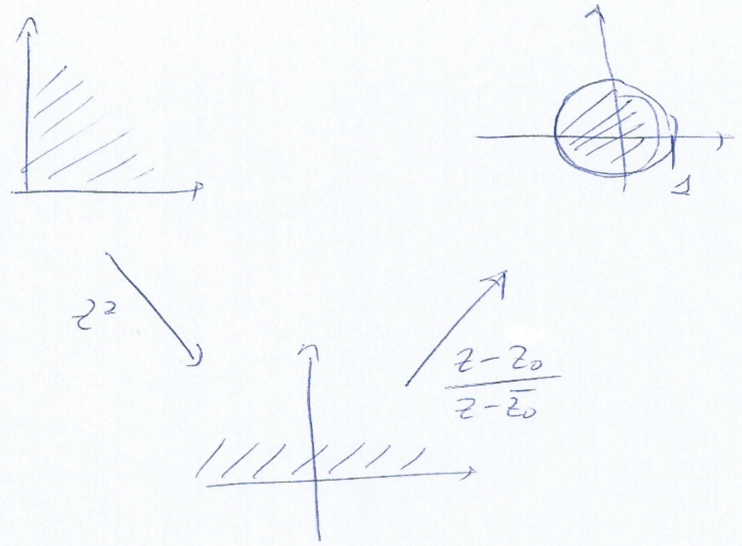
$$= \frac{e^{x_0}\sin y_0}{\pi}\cdot\frac{1}{\sqrt{c^2-b^2}}\cdot \operatorname{arctg}\frac{t-b}{\sqrt{c^2-b^2}}\Big|_0^\infty = \left\{ \begin{matrix} \sqrt{c^2-b^2} = \sqrt{e^{2x_0}-e^{2x_0}\cos^2 y_0} = \\ = e^{x_0}|\sin y_0| \\ 0<y<\pi \Rightarrow \sin y_0 \geq 0 \\ \Rightarrow |\sin y_0| = \sin y_0 \end{matrix} \right\} =$$

$$= \frac{e^{x_0}\sin y_0}{\pi\, e^{x_0}\sin y_0}\operatorname{arctg}\frac{t-e^{x_0}\cos y_0}{e^{x_0}\sin y_0}\Big|_0^\infty = \frac{1}{\pi}\left(\frac{\pi}{2} - \operatorname{arctg}\left(-\frac{e^{x_0}\cos y_0}{e^{x_0}\sin y_0}\right)\right) =$$

$$= \frac{1}{2} + \frac{1}{\pi}\operatorname{arctg}(\operatorname{ctg} y_0) = \frac{1}{2} + \frac{1}{\pi}\left(\frac{\pi}{2} - y_0\right) = 1 - \frac{y_0}{\pi}$$

№3 Д3 7

$$\begin{cases} \Delta u = 0 & x>0,\ y>0 \\ u(0,y) = a \\ u(x,0) = b \end{cases}$$



$$f = \frac{z^2 - z_0^2}{z^2 - \bar z_0^2}$$

$$G(z_0, z) = \frac{1}{2\pi} \ln \frac{1}{|f(z_0,z)|}$$

$$u(x_0,y_0) = -\int_\Gamma \frac{\partial G}{\partial n}\Big|_\Gamma \varphi\, d\Gamma =$$

$$= -\int_0^\infty \frac{\partial G}{\partial(-x)}\Big|_{x=0} a\, dy - \int_0^\infty \frac{\partial G}{\partial(-y)}\Big|_{y=0} b\, dx = a\int_0^\infty \frac{\partial G}{\partial x}\Big|_{x=0} dy + b\int_0^\infty \frac{\partial G}{\partial y}\Big|_{y=0} dx$$

$$|f(z_0,z)| = \frac{A_1}{A_2}$$

$$A_1 = |z^2 - z_0^2| = |(x+iy)^2 - (x_0+iy_0)^2| = |x^2 + 2xiy - y^2 - x_0^2 - 2x_0 i y_0 + y_0^2| =$$

$$= \left((x^2-y^2-x_0^2+y_0^2)^2 + (2xy - 2x_0y_0)^2\right)^{1/2}, \quad A_1\Big|_{y=0} = \left((x^2-x_0^2+y_0^2)^2 + (2x_0y_0)^2\right)^{1/2}$$

$$A_2 = \left((x^2-y^2-x_0^2+y_0^2)^2 + (2xy+2x_0y_0)^2\right)^{1/2}, \quad A_2\Big|_{y=0} = c$$

$$\frac{\partial A_1}{\partial y}\Big|_{y=0} = \frac{1}{2A_1}\cdot\left(2(x^2-y^2-x_0^2+y_0^2)(-2y) + 2(2xy-2x_0y_0)\cdot(2x)\right)\Big|_{y=0} = \frac{-4x_0y_0x}{A_1|_{y=0}} = \frac{-4x_0xy_0}{c}$$

$$\frac{\partial A_2}{\partial y}\Big|_{y=0} = \frac{4x_0xy_0}{c}$$

$$G(z_0,z) = \frac{1}{2\pi}\ln\left|\frac{1}{f(z_0,z)}\right| = \frac{1}{2\pi}\ln\frac{A_2}{A_1}$$

$$\frac{\partial G}{\partial y}\Big|_{y=0} = \frac{1}{2\pi}\frac{A_1}{A_2}\cdot\left(\frac{A_2}{A_1}\right)'\Big|_{y=0} = \frac{A_1}{2\pi A_2}\cdot\frac{A_2'A_1 - A_2A_1'}{A_1^2}\Big|_{y=0} = \frac{A_2'A_1 - A_2A_1'}{2\pi A_1A_2}\Big|_{y=0} = \frac{4x_0xy_0}{\pi c^2} = \frac{4x_0xy_0}{\pi\left((x^2-x_0^2+y_0^2)^2 + (2x_0y_0)^2\right)}$$

По x аналогично, только где-то могут поменяться знаки

Судя по всему, там и интегралы неплохо считаются

$$\int_0^\infty \frac{\partial G}{\partial y}\Big|_{y=0} dx \sim \int_0^\infty \frac{x\,dx}{(x^2+c)^2+\tilde c} = \int_0^\infty \frac{dt}{(t+c)^2+\tilde c}$$ — тот же самый арктангенс вроде,

но в задаче просят только формальное решение